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# *An Expression of the Coordinates of a Point on a Bi-Nodal Quartic Curve as Rational Functions of the Elliptic Functions of a Variable Parameter.*

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The equation of the bi-nodal quartic can always be put in the form

$$(1) \qquad x^2 y^2 + z^2 (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0,$$

where $z = 0$ is the line joining the nodes and $x = 0$ and $y = 0$ its harmonic conjugates with regard to the tangents at the nodes.

We may otherwise write (1)

$$\frac{y^2}{z^2}\left(\frac{x^2}{z^2} + b\right) + 2\frac{y}{z}\left(f + h\frac{x}{z}\right) + \left(a\frac{x^2}{z^2} + 2g\frac{x}{z} + c\right) = 0.$$

This gives

$$(2) \qquad \left(\frac{x^2}{z^2} + b\right)\frac{y}{z} =$$

$$-f - h\frac{x}{z} \pm \sqrt{-\left(a\frac{x^4}{z^4} + 2g\frac{x^3}{z^3} + (c + ab - h^2)\frac{x^2}{z^2} + 2(bg - hf)\frac{x}{z} + bc - f^2\right)}.$$

Suppose that $\alpha$, $\beta$, $\gamma$, $\delta$ are the roots of the quartic in $\frac{x}{z}$ under the radical sign, and let $\alpha < \beta < \gamma < \delta$.

Assume

$$(3) \qquad \frac{x}{z} = \frac{\alpha(\beta - \delta) + \delta(\alpha - \beta)s^2}{\beta - \delta + (\alpha - \beta)s^2} = \frac{A + Bs^2}{C + Ds^2},$$

where $s^2$ is a new variable.

Also let $$k^2 = \frac{\alpha - \beta . \gamma - \delta}{\alpha - \gamma . \beta - \delta},$$

the quartic in $\frac{x}{z}$ then becomes $-a \cdot \frac{x}{z} - \alpha \cdot \frac{x}{z} - \beta \cdot \frac{x}{z} - \gamma \cdot \frac{x}{z} - \delta$,

which is $$a\frac{(\alpha - \beta)^2 (\alpha - \gamma)(\alpha - \delta)^2 (\beta - \delta)^3 . s^2 (1 - s^2)(1 - k^2 s^2)}{[(\beta - \delta) + (\alpha - \beta)s^2]^4},$$

This vanishes for $\frac{x}{z} = \alpha, \beta, \gamma, \delta$, that is for $s^2 = 0, 1, \frac{1}{k^2}, \infty$.

By reference to (2) we see that $x=\alpha z$, $x=\beta z$, $x=\gamma z$, $x=\delta z$ are the four tangents from the node $(xz)$ to the curve. $k^2$ is therefore one of the anharmonic ratios of these tangents.

Put
$$\sqrt{a(\alpha-\beta)^2(\alpha-\gamma)(\alpha-\delta)^2(\beta-\delta)^3}=n$$
$$s^2,\ 1-s^2,\ 1-k^2s^2=\text{sn}^2u,\ \text{cn}^2u,\ \text{dn}^2u=s^2,\ c^2,\ d^2,$$
$$A+Bs^2=M,\quad C+Ds^2=N,\quad scd=P,$$
(2) and (3) then give
$$(M^2+bN^2)\frac{y}{z}=-fN^2-hMN\mp nP,$$
or
$$x:y:z=M(M^2+bN^2):-N(fN^2+hMN\pm nP):M(M^2+bN^2).$$
The double sign before the $nP$ is to be taken as positive or negative according as we take one or the other of the two intersections of a line through $(xz)$ with the curve.

The ratio $x:y:z$, as above given, involves nine constants, $A$, $B$, $C$, $D$, $b$, $f$, $h$, $k^2$, $n$, or the 8 ratios of these constants, which is two more than are needed. Of these $n$ involves $a$ and $\alpha$, $\beta$, $\gamma$, $\delta$; $a$ and likewise $b$ can be expressed in terms of $h$, $f$, and $\alpha$, $\beta$, $\gamma$, $\delta$; and then $\alpha$, $\beta$, $\gamma$, $\delta$ themselves, in terms of $A$, $B$, $C$, $D$ and $k^2$.

We have
$$a=-\frac{2g}{\Sigma\alpha}=-\frac{2(bg-hf)}{\Sigma\alpha\beta\gamma}=\frac{2hf}{\Sigma\alpha\beta\gamma-b\Sigma\alpha},$$
also
$$a=\frac{c+ab-h^2}{\Sigma\alpha\beta}=\frac{bc-f^2}{\alpha\beta\gamma\delta}=\frac{c-h^2}{\Sigma\alpha\beta-b}=\frac{bh^2-f^2}{\alpha\beta\gamma\delta-b\Sigma\alpha\beta+b^2},$$
whence
$$0=b^2(2hf+h^2\Sigma\alpha)-b(f^2\Sigma\alpha+2hf\Sigma\alpha\beta+h^2\Sigma\alpha\beta\gamma)+f^2\Sigma\alpha\beta\gamma+2hf\alpha\beta\gamma\delta,$$
where $\alpha$, $\beta$, $\gamma$, $\delta$ are to be replaced by their values in terms of $A$, $B$, $C$, $D$, $k^2$, viz.
$$\alpha=\frac{A}{C},\quad \beta=\frac{A+B}{C+D},\quad \gamma=\frac{k^2A+B}{k^2C+D},\quad \delta=\frac{B}{D}.$$
The value of $x:y:z$ readily gives for the node $(xz)$
$$s^2=-\frac{A\pm\sqrt{-b}\,C}{B\pm\sqrt{-b}\,D}=-\frac{\alpha\pm\sqrt{-b}}{\delta\pm\sqrt{-b}}\cdot\frac{C}{D},$$
and for the node $(yz)$
$$s^2=-\frac{C}{D}.$$

And since the tangents at the nodes are $x^2+bz^2=0$, and $y^2+az^2=0$, we have for the tangentials of the node $(xz)$
$$s^2=-\frac{A\pm\sqrt{-b}\,C}{B\pm\sqrt{-b}\,D},$$
the same values as for the node itself.

Those of the node $(yz)$ are given by

$$fN^2+hMN \pm nP = \pm\,(M^2+bN^2)\sqrt{-a}.$$

If a line through $(xz)$ meet the quartic in the points whose parameters are $u$ and $u_1$, then if $s$, $c$, $d$ and $s_1$, $c_1$, $d_1$ are the sn, cn, and dn of those parameters, $M$, $N$, $P$, $M_1$, $N_1$, $P_1$, the corresponding functions of $s$, $c$, $d$ and $s_1$, $c_1$, $d_1$, we have

$$0=\begin{vmatrix} x, & z \\ x_1, & z_1 \end{vmatrix} = \begin{vmatrix} M, & N \\ M_1, & N_1 \end{vmatrix} = \begin{vmatrix} A+Bs^2, & C+Ds^2 \\ A+Bs_1^2, & C+Ds_1^2 \end{vmatrix}$$
$$=(BC-AD)(s^2-s_1^2),$$

and therefore $$s=\pm\,s_1,$$
an equation which we have already seen to hold with regard to the tangents at the node $(xz)$. We shall presently show that $s=s_1$ and $s=-s_1$ are true for lines cutting different portions of the curve, the change from one to the other occurring at the points where $s=0$ and $s=\infty$.

The line through the node $(yz)$ is found in precisely the same way to meet the quartic in points whose parameters are connected by the relation

$$0=\begin{vmatrix} -y, & z \\ -y_1, & z_1 \end{vmatrix} = \begin{vmatrix} fN^2+hMN \pm nP, & M^2+bN^2 \\ fN_1^2+hM_1N_1 \pm nP_1, & M_1^2+bN_1^2 \end{vmatrix}$$
$$=f\begin{vmatrix} N^2, & M^2 \\ N_1^2, & M_1^2 \end{vmatrix} - h\,(MM_1-bNN_1)\begin{vmatrix} M, & N \\ M_1, & N_1 \end{vmatrix} \pm n\begin{vmatrix} P, & M^2+bN^2 \\ P_1, & M_1^2+bN_1^2 \end{vmatrix}$$
$$=(s^2-s_1^2)(BC-AD)\left[\begin{matrix}[2AC+2BDs^2s_1^2+(BC+AD)(s^2+s_1^2)]f \\ -h\,(A+Bs^2)(A+Bs_1^2)+hb\,(C+Ds^2)(C+Ds_1^2)\end{matrix}\right.$$
$$\pm\, n\left[scd\left((A+Bs_1^2)^2+b\,(C+Ds_1^2)^2\right)-s_1c_1d_1\left((A+Bs^2)^2+b\,(C+ds^2)^2\right)\right].$$

This result may be easily verified for the line $z=0$ meeting the quartic in the node $(xz)$ counting as two points for which

$$s=\sqrt{\frac{A+\sqrt{-b}\,C}{B+\sqrt{-b}\,D}},\quad s_1=\sqrt{\frac{A-\sqrt{-b}\,C}{B-\sqrt{-b}\,D}}.$$

On substituting these values the quantities multiplying $f$, $h$ and $n$ separately vanish.

Again, if we put $s_1^2=-\dfrac{C}{D}$, the value of $s^2$ is that for the tangentials of the node $(yz)$. We get

$$\frac{fN^2+hMN \pm nP}{M^2+bN^2}=\pm\,\frac{nP_1}{M_1^2}=\pm\sqrt{-a},$$

which is the same as the equation previously obtained for these tangentials.

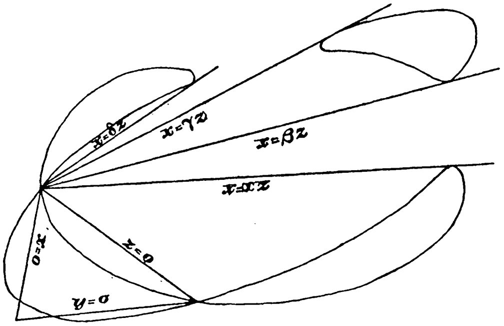
Consider again a line through the node $(xz)$. We had for such a line $s = \pm s_1$; and if the line $z$ were tangent to the quartic, $s$ assumed the values $0,\ 1,\ \frac{1}{k}$, and $\infty$ as $\frac{x}{z}$ became equal to $\alpha$, $\beta$, $\gamma$, and $\delta$ successively.

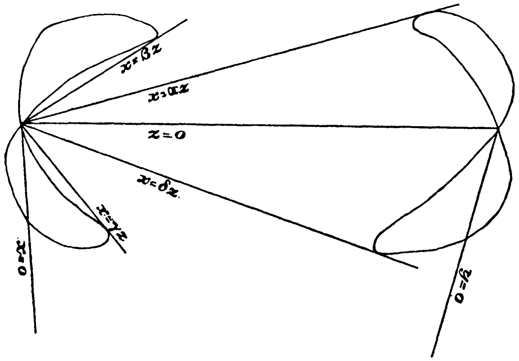
If in (3) we put $\frac{x}{z} > \delta$ or $\frac{x}{z} < \alpha$, we find $s$ a pure imaginary; the argument $u$ is therefore imaginary or differs from a pure imaginary by a half-period. We must then for this portion of the curve have $s = -s_1$ and argument varying from $u = \pm iK'$ at $s = \infty$, to $u = 0$ at $s = 0$. For intermediate points we have $u = +i\lambda$ and $u = -i\lambda$.

Now since for $x = \beta z$ $s = 1$, it must be that for $x = \beta z \pm d$, where $d$ is an infinitesimal, $s$ varies but little from 1; therefore both values of $s$ must be positive, *i. e.* $s = s_1$: the same reasoning shows that $s$ continues equal to $s_1$ until we come to $s = \infty$ again.

For $s = 1$ we have $u = K$, also for $s = \frac{1}{k}$, $u = K \pm iK'$, and we are enabled to see how $u$ varies as the line through $(xz)$ revolves. From $x = \alpha z$, where $u = 0$ or $2K$, to $x = \beta z$, $u = K \pm \lambda$ becoming $K$ at $x = \beta z$; from $x = \beta z$ to $x = \gamma z$, $u = K \pm i\lambda$ becoming $K \pm iK'$ at $x = \gamma z$; finally, from $x = \gamma z$ to $x = \delta z$, $u = K \pm iK' \pm \lambda$ becoming $iK'$ at $x = \delta z$.

There are two bi-nodal quartics, the tangents at the nodes of which and the four tangents from the nodes of which are real. In one the two nodes are on the same branch of the curve, and in the other on different branches of the curve.





For the first, which I have sketched to the left, $\alpha$, $\beta$, $\gamma$, $\delta$ all have the same sign. In the drawing I have assumed all negative, since then $x$ and $z$ will both be positive when both are reckoned the same way from the centre of the triangle of reference. For the second form of the curve $\alpha$ and $\beta$ are negative, and $\gamma$ and $\delta$ positive. In either case $(yz)$ lies upon that portion of the curve for which the argument and therefore the $s$ is imaginary. This is as it should be, for the $s$ at $(xz)$ is $\sqrt{-\frac{C}{D}}$, and $\frac{C}{D}$ is always positive. In the first case $(xz)$ lies upon the portion with imaginary $s$, in the second case not. This is also as it should be. For in the case where $s$ of $(xz)$ is imaginary, either $\alpha$ and $\delta$ are negative and $-b > \delta^2$ or $\alpha$ and $\delta$ are positive and $1 - b < \alpha^2$, so that we must have $\frac{\alpha \pm \sqrt{-b}}{\delta \pm \sqrt{-b}}$ positive, and therefore

$$\sqrt{-\frac{\alpha \pm \sqrt{-b}C}{\delta \pm \sqrt{-b}D}} = \sqrt{-\frac{A \pm \sqrt{-b}C}{B \pm \sqrt{-b}D}} \text{ is imaginary.}$$

When, on the other hand, $s$ of $(xz)$ is real, $\alpha$ is negative, $\delta$ positive, $-b < \alpha^2$ and $< \delta^2$, giving $\frac{\alpha \pm \sqrt{-b}}{\delta \pm \sqrt{-b}}$ negative and so $\sqrt{-\frac{A \pm \sqrt{-b}C}{B \pm \sqrt{-b}D}}$ is real.

It is obvious, that instead of solving the original equation as one in $\frac{y}{z}$ we could have solved it as one in $\frac{x}{z}$ by putting

$$\frac{y}{z} = \frac{M'}{N'} = \frac{A' + B'\sigma^2}{C' + D'\sigma^2} = \frac{\alpha'(\beta' - \delta') + \delta'(\alpha' - \beta')\sigma^2}{\beta' - \delta' + (\alpha' - \beta')\sigma^2},$$

$y = \alpha' z$, $\beta' z$, $\gamma' z$, $\delta' z$ being the tangents from the node $(yz)$ to the curve and $\alpha' < \beta' < \gamma' < \delta'$. All that we have made out with regard to a line through $(xz)$ now applies equally to a line through $(yz)$, if we change the unaccented for accented letters and the $b$ for $a$, $a$ for $b$, and $f$ for $g$.

The $k^2$ remains unchanged since the anharmonic ratio of the four tangents from $(yz)$ = that of the four from $(xz)$. The parameters for the tangents from $(yz)$ have then the values $0$, $K$, $K + iK'$, $iK'$. The intersections of corresponding tangents lie on a conic through the nodes. If we denote by $v$ the parameter of the points of the quartic when we put $\frac{y}{z} = \frac{M'}{N'}$, then $u = v$ means that the line through $(xz)$ and $u$ (say) meets the line through $(yz)$ and $v$ on the above mentioned conic. The condition that lines through $(xz)$ and $(yz)$ should so meet is that they are corresponding rays of the two homographic pencils determined by the tangents from the two nodes. Therefore we have, expressing this condition analytically by writing the general values of $\frac{x}{z}$ and $\frac{y}{z}$ instead of $\alpha$ and $\alpha'$ in the values of $k^2$ and by equating results,

$$\frac{\left(\frac{\alpha(\beta - \delta) + \delta(\alpha - \beta)s^2}{\beta - \delta + (\alpha - \beta)s^2} - \beta\right)(\gamma - \delta)}{\left(\frac{\alpha(\beta - \delta) + \delta(\alpha - \beta)s^2}{\beta - \delta + (\alpha - \beta)s^2} - \gamma\right)(\beta - \delta)} = \frac{\left(\frac{\alpha'(\beta' - \delta') + \delta'(\alpha' - \beta')\sigma^2}{\beta' - \delta' + (\alpha' - \beta')\sigma^2} - \beta'\right)(\gamma' - \delta')}{\left(\frac{\alpha'(\beta' - \delta') + \delta'(\alpha' - \beta')\sigma^2}{\beta' - \delta' + (\alpha' - \beta')\sigma^2} - \gamma'\right)(\beta' - \delta')}.$$

This gives immediately

$$\frac{(\alpha-\beta)(\beta-\delta)(1-s^2)(\gamma-\delta)}{(\alpha-\gamma)(\beta-\delta)+(\delta-\gamma)(\alpha-\beta)s^2(\beta-\delta)}=\frac{(\alpha'-\beta')(\beta'-\delta')(1-\sigma^2)(\gamma'-\delta')}{(\alpha'-\gamma')(\beta'-\delta')+(\delta'-\gamma')(\alpha'-\beta')\sigma^2(\beta'-\delta')},$$

$$\text{or}\quad \frac{k^2(1-s^2)}{1-k^2s^2}=\frac{k^2(1-\sigma^2)}{1-k^2\sigma^2}\quad \textit{i. e. } s^2=\sigma^2.$$

It was not at all necessary to have assumed $\alpha<\beta<\gamma<\delta$; we might have taken them in any order whatsoever and have changed them in 4 different ways without altering the $k^2$. The same is of course true of the $\alpha'$, $\beta'$, $\gamma'$ and $\delta'$. Corresponding to a definite value of $k^2$ there will then be 4 different conics through the nodes on which corresponding lines through the nodes will meet.